Propagation of Light in Cantor Media

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We numerically find that transmission coefficients have a rich structure as a function of wavelength in Cantor media. Complete transmission and complete reflection are observed. We also find that light propagation has scalings with respect to number of layers.

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Localization of electronic states due to disorder is one of the most active fields in condensed-matter physics [1]. It has been widely recognized that localization could occur not only in disordered systems but also in the quasiperiodic systems in one dimension [2]. In a quasiperiodic system two (or more) incommensurate periods are superposed, so that it is neither a periodic nor a random system and could be considered to be intermediate between the two.

In one dimension, a quasiperiodic Schrödinger equation based on the Fibonacci sequence has been analyzed by a renormalization-group type theory [3, 4]. In this model, a simple binary quasiperiodic sequence is used which is constructed recursively as $F_{j+1} = \{F_{j-1}, F_j\}$, for $j \geq 1$, with $F_0 = \{B\}$ and $F_1 = \{A\}$. In this sequence one has $F_2 = \{BA\}$, $F_3 = \{ABA\}$, $F_4 = \{BAABA\}$, and so on. The most striking feature of this model is that all the states are critical. Namely, the wave functions are not localized exponentially but only weakly localized and have a rich structure including scaling [4, 5]. The energy spectrum also has a rich structure. It was found to have zero Lebesgue measure. Namely, if one picks an energy, it is in a gap with probability one and the gaps are dense. Also there are no isolated points. The spectrum has a self-similar structure with various scaling indices (multifractal) [5, 6].

While the localization of states was originally regarded as an electronic problem, it was later recognized that the phenomenon is essentially a consequence of the wave nature of the electronic states. Therefore, such localization can be expected for any wave phenomenon. An optical experiment with the Fibonacci layer was proposed [7]. The corresponding experiments were reported [8, 9] using a dielectric multilayer stacks of SiO_2 and TiO_2 thin films. See also [10]. The transmission coefficients and the scaling properties remarkably agree with the theory [7] and can be considered as an experimental evidence for localization of light waves. Recently, an experiment for propagation of electromagnetic wave in a three-dimensional fractal cavity was reported [11]. The material is called the Menger sponge and the structure is the cubic Sierpinski gasket which has a self-similar pattern with single scaling index (not multifractal). The results are interpreted as a localization of the electromagnetic waves [11].

In this paper, we propose an optical experiment with Cantor layers. In this system one-dimensional theory is strictly valid. Also, it is feasible to construct a system accurately and the parameters are precisely controlled and measured. We calculate the transmission coefficient as a function of wavelength of light. The results show a singular structure, i.e. it alternates between complete transmission and complete reflection. The presence of complete reflection is quite striking, because the substrate of the Cantor layer has zero Lebesgue measure in the thermodynamic limit of the generation. Namely, if one picks the substrate, it is in vacuum with probability one and the vacuum is dense. Also there are no isolated points. We also find that the light propagation has scaling with respect to the number of layers.

The procedure of constructing the Cantor set begins with a line segment of unit length. We regard this as substrate A. This line segment is divided into three equal parts and the middle part is removed to obtain the first generation; that serves as the "generator" of the Cantor set. The removed area is substrate B. The procedure is repeated for each of the two line segments of the first generation to obtain the second generation and so on. Therefore the j-th generation of the Cantor set is a finite set of 2^j line segments, each of length $1/3^j$. If this procedure is repeated an infinite number of times the remainder set of discrete points is called the Cantor set. It is an exact self-similar fractal of dimension $\log_3 2$, which
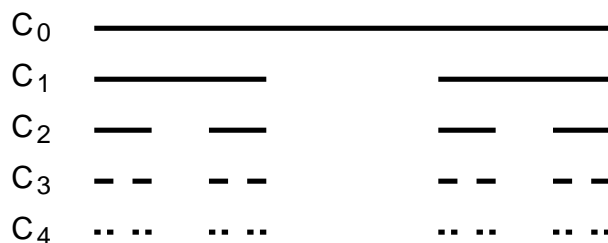

FIG. 1: The initial condition $j = 0$ and the first four generations in the construction of the Cantor set. The first generation, i.e. $n = 1$, provides a "generator." We call the j-th generation a Cantor sequence C_j.

is a single scaling dimension and is not a multi-fractal. The construction of first few generation is shown in Fig.1.

Let us consider a multilayer in which two types of layers A and B are arranged in a Cantor sequence. In order to understand the light propagation in this media, first consider an interface of two layers. See Fig.2. The electric field for the light in layer A is given by

$$\vec{E} = \vec{E}_A^{(1)} e^{i(\vec{k}_A^{(1)} \cdot \vec{x} - \omega t)} + \vec{E}_A^{(2)} e^{i(\vec{k}_A^{(2)} \cdot \vec{x} - \omega t)}. \qquad (1)$$

The electric field in layer B is given by the same expression with subscript A replaced by B. We consider a polarization which is perpendicular to the plane of the light path (TE wave). The appropriate boundary condition at an interface gives

$$E_A^{(1)} + E_A^{(2)} = E_B^{(1)} + E_B^{(2)},$$
$$n_A \cos\theta_A (E_A^{(1)} - E_A^{(2)}) = n_B \cos\theta_B (E_B^{(1)} - E_B^{(2)}). (2)$$

where n_A and n_B are indices of refraction of A and B, respectively, and the angles θ_A and θ_B are shown in Fig.2. Snell's law is $\sin\theta_A / \sin\theta_B = n_B/n_A$. It is convenient to choose the two independent variables for the light as

$$E_+ = E^{(1)} + E^{(2)}, \quad E_- = (E^{(1)} - E^{(2)})/i. \qquad (3)$$

Then (2) gives

$$\begin{bmatrix} E_+ \\ E_- \end{bmatrix}_B = T_{BA} \begin{bmatrix} E_+ \\ E_- \end{bmatrix}_A, \qquad (4)$$

where T_{BA} is given by

$$T_{BA} = \begin{bmatrix} 1 & 0 \\ 0 & n_A \cos\theta_A / n_B \cos\theta_B \end{bmatrix}. \qquad (5)$$

Also we define

$$T_{AB} = T_{BA}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & n_B \cos\theta_B / n_A \cos\theta_A \end{bmatrix}. \qquad (6)$$
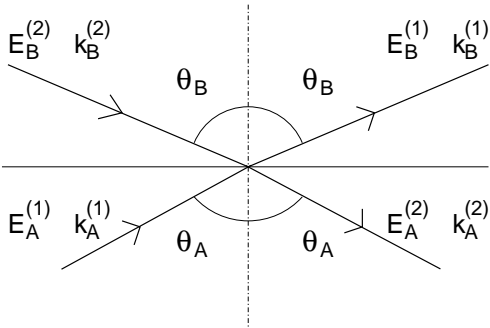

FIG. 2: Electromagnetic wave propagation across an interface of two layers A and B.

The matrices T_{BA} and T_{AB} represent the light propagation across interfaces $B \leftarrow A$ and $A \leftarrow B$, respectively. The propagation within one layer is represented by

$$T_A(d_A) = \begin{bmatrix} \cos\delta_A(d_A) & -\sin\delta_A(d_A) \\ \sin\delta_A(d_A) & \cos\delta_A(d_A) \end{bmatrix}, \qquad (7)$$

for a layer of type A where d_A is the thicknesses of the layer, and the same expression for $T_B(d_A)$ in which
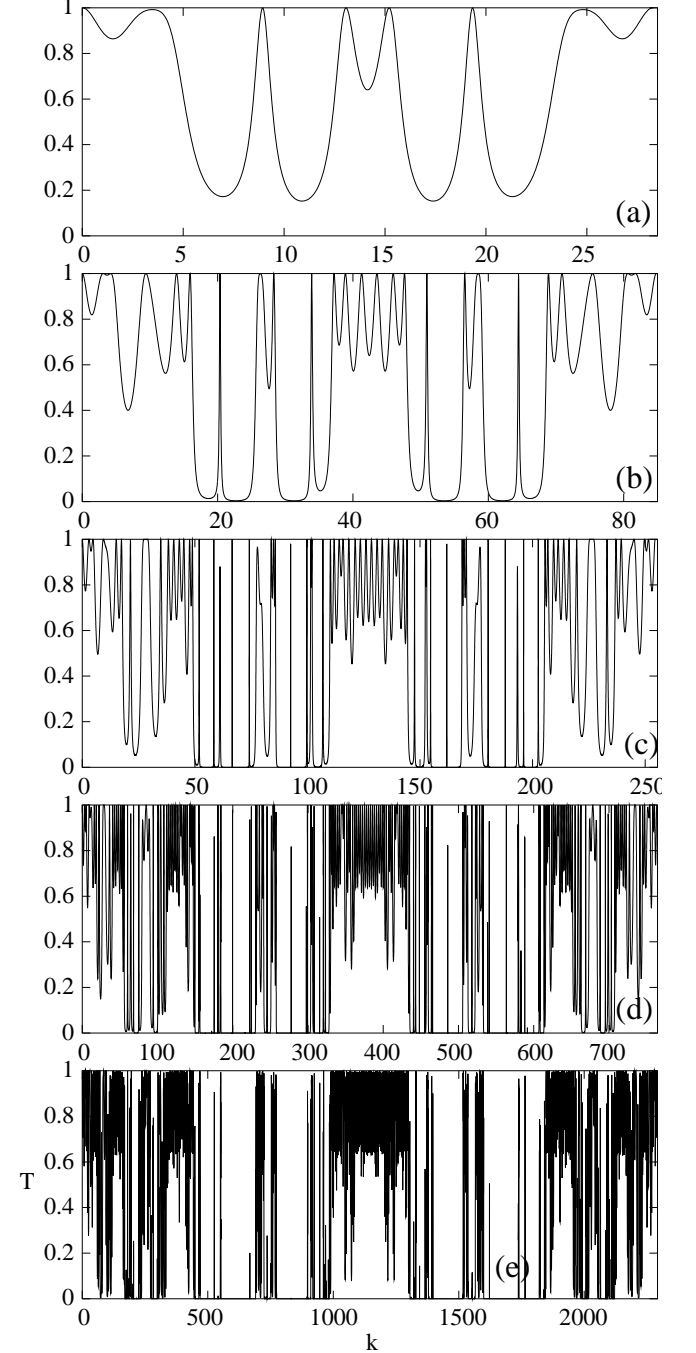

FIG. 3: Transmission coefficient T as a function of the wave number k for multilayers C_2 (a), C_3 (b), C_4 (c), C_5 (d), and C_6 (e).

$\delta_A(d_A)$ is replaced by $\delta_B(d_B)$. The phases are given by

$$\delta_A(d_A) \;=\; n_A k d_A / \cos\theta_A,$$

and

$$\delta_B(d_B) \;=\; n_B k d_B / \cos\theta_B, \tag{8}$$

where k is the wave number in vacuum.

Now we are ready to consider light propagation through a Cantor multilayer C_j which is sandwiched by two media of material of type A. For zero generation layer $A(d_A)$ and 1st generation layers $A(d_A/3)B(d_A/3)A(d_A/3)$, the light propagation are respectively given by

$$M_0 \;=\; T_A(d_A),$$
$$M_1 \;=\; T_A\!\left(\frac{d_A}{3}\right) T_{AB} T_B\!\left(\frac{d_A}{3}\right) T_{BA} T_A\!\left(\frac{d_A}{3}\right). \tag{9}$$

It can be shown that for j-th generation, i.e. C_j, the corresponding matrix M_j is obtained by recursive replacement of

$$T_A\!\left(\frac{d_A}{3^{j-1}}\right), \tag{10}$$

in M_{j-1} by

$$T_A\!\left(\frac{d_A}{3^j}\right) T_{AB} T_B\!\left(\frac{d_A}{3^j}\right) T_{BA} T_A\!\left(\frac{d_A}{3^j}\right). \tag{11}$$

The transmission coefficient T is given in terms of the matrix M_j as

$$T = \frac{4}{|M_j|^2 + 2} \tag{12}$$

where $|M_j|^2$ is the sum of the squares of the four elements of M_j. This is a quantity measured experimentally and has a rich structure with respect to a variation of either the wavelength of the light or the number of layers.

Let us consider the simplest experimental setting. Take the incident light to be normal, (i.e. $\theta_A = \theta_B = 0$) and also choose the layer B is in vacuum, $n_B = 1$. The results for transmission are shown in Fig.3 for Cantor sequences C_2 to C_6 where we set $n_A = 2$. The global structure of the transmission coefficient apparently show a scaling. Namely, as the generation of the Cantor sequence is increased, fine structures appear in addition to the former one. We also find a binary structure between complete transmission and complete reflection as a function of the wave number as the generation is increased. This behavior is distinct from that of the Fibonacci case [7]. The Cantor set under consideration has zero Lebesgue measure (in the thermodynamic limit). This means that, if one points the substrate, it is in vacuum, i.e. substrate B, with probability one and the vacuum is dense. Also there are no isolated points. The complete reflection is quite striking, because the above means that if one points the substrate, one points the substrate A with probability zero. Also the complete transmission has to be understood. So far we do not have any appropriate explanation for this novel behavior: complete transmission and complete reflection.

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